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Killing Spinor Equations from Nonlinear Realisations

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Abstract

Starting from a nonlinear realisation of eleven dimensional supergravity based on the group G_{11}, whose generators appear as low level generators of E_{11}, we present a super extended algebra, which leads to a covariant derivative of spinors identical to the Killing spinor equation of this theory. A similar construction leads to the Killing spinor equation of $N = 1$ pure supergravity in ten dimensions.

1 Introduction

The search for hidden symmetries of eleven dimensional supergravity (M-theory) [1][*1] has a long history [2]. In the context of supergravities' relation to string theories the existence of hidden symmetries realising U-duality was first conjectured in [3]. Identifying the (hidden) symmetry group of a given supergravity theory is of vast interest for understanding its properties; it is essential for finding solution generating techniques, but also for pinpointing the whole issue of dualities that apparently relate different supergravities to each other.

Several approaches to manifest these extra symmetries have been developed in the past. In [4, 5] it was shown that the non-gravitational degrees of freedom of almost all supergravity theories can be described as a non-linear realisation which made part of the hidden symmetries manifest. The result was obtained by using the doubled field method. The additional degrees of freedom introduced by doubling of the fields are projected out by the equations of motion of the gauge fields which take the form of twisted selfduality conditions. The generators of this coset construction were inert under Lorentz transformations, and as such it is difficult to extend this method straightforwardly to include gravity or fermions. In [6] it was shown that the entire bosonic sector of eleven- and ten dimensional IIA supergravity could be formulated as a non-linear realisation. In this way of proceeding, gravity is treated on an equal footing with the gauge fields and thus is naturally built in. The method of non-linear realisations has consequently been shown to extend to other gravity theories [7, 8, 9]. However, it did not include the fermionic degrees of freedom.

The non-linear realisation of M-theory of [6] is based on the nonsimple algebra G_{11} with group element

$$g_B \;=\; e^{x^\mu P_\mu}\, e^{\kappa_a{}^b K^a{}_b}\, \exp\left(\frac{1}{3!}\, A_{c_1\ldots c_3} R^{c_1\ldots c_3} \;+\; \frac{1}{6!}\, A_{c_1\ldots c_6} R^{c_1\ldots c_6}\right) \tag{1.1}$$

that was shown to generate the covariant structure of the bosonic fields (subscript B) and their equations of motion. The group G_{11} is defined by the algebra (only non-trivial commutators displayed)

$$[\, K^a{}_b,\, K^c{}_d\,] \;=\; \delta^c_b\, K^a{}_d \;-\; \delta^a_d\, K^c{}_b \qquad\qquad [\, K^a{}_b,\, P_c\,] \;=\; -\,\delta^a_c\, P_b \tag{1.2}$$

$$[\, K^a{}_b,\, R^{c_1\ldots c_3}\,] \;=\; 3\,\delta^{[c_1}_b\, R^{c_2 c_3]a} \qquad\qquad [\, K^a{}_b,\, R^{c_1\ldots c_6}\,] \;=\; -\,6\,\delta^{[c_1}_b\, R^{c_2\ldots c_6]a} \tag{1.3}$$

$$[\, R^{c_1\ldots c_3},\, R^{c_4\ldots c_6}\,] \;=\; c_{3,3}\, R^{c_1\ldots c_6} \tag{1.4}$$

and the Cartan form of the coset ($h_{ab} = \kappa_{(ab)}$, $\kappa_{[ab]} = 0$) reads

$$g_B^{-1} dg_B \;=\; dx^\mu \left\{\, P_\mu \;+\; \left(e^{-h}\,\partial_\mu\, e^h\right)_a{}^b K^a{}_b \;+\; \frac{1}{4!}\, \left(4\,\tilde{D}_{[\mu} A_{c_1 c_2 c_3]}\right) R^{c_1 c_2 c_3} \;+\; \ldots \right. \tag{1.5}$$

$$\left. \ldots \;+\; \frac{1}{7!}\, \left(7\,\tilde{D}_{[\mu} A_{c_1\ldots c_6]}\right) R^{c_1\ldots c_6}\, \right\}$$

with

$$\tilde{D}_\mu A_{c_1 c_2 c_3} \;=\; \partial_\mu A_{c_1 c_2 c_3} \;+\; \left(\left(e^{-h}\,\partial_\mu\, e^h\right)_{c_1}{}^b A_{b c_2 c_3} \;+\; \left(e^{-h}\,\partial_\mu\, e^h\right)_{c_2}{}^b A_{c_1 b c_3} \;+\; \left(e^{-h}\,\partial_\mu\, e^h\right)_{c_3}{}^b A_{c_1 c_2 b}\right)$$

$$\tilde{D}_\mu A_{c_1\ldots c_6} \;=\; \partial_\mu A_{c_1\ldots c_6} \;+\; \left(\left(e^{-h}\,\partial_\mu\, e^h\right)_{c_1}{}^b A_{b c_2\ldots c_6} \;+\; \left(e^{-h}\,\partial_\mu\, e^h\right)_{c_2}{}^b A_{c_1 b c_3\ldots c_6} \;+\; \ldots\right.$$

$$\left. \ldots \;+\; \left(e^{-h}\,\partial_\mu\, e^h\right)_{c_6}{}^b A_{c_1\ldots c_5 b}\right) \;-\; 10\, A_{[c_1\ldots c_3}\, \tilde{D}_{|\mu|} A_{c_4\ldots c_6]}\, c_{3,3}\;.$$

[*1] Our conventions differ from those in [1] by rescaling all gauge fields by a factor of $1/2$, working with a mostly plus signature and replacing Γ^a by $i\,\Gamma^a$.

The antisymmetry of the indices indicated in (1.5) is not obtained automatically by the procedure outlined so far but it is the result of a second step, *i.e.* making the Cartan form of G_{11} simultaneously covariant with respect to the conformal group [10, 6]. Identifying the resulting objects with the two field strengths $G_{a_1...a_4}$ and $F_{a_1...a_7}$ the equation of motion of the gauge field of M-theory reads

$$* G_{(4)} \quad = \quad F_{(7)} \ . \tag{1.6}$$

In this way the bosonic gauge sector of eleven dimensional supergravity is completely described by the covariant field strengths together with a geometric equation of motion. The second term in eq. (1.5) requires special attention. From the transformation properties of the Cartan form according to the transformation $g_B \mapsto g \cdot g_B \cdot h^{-1}$ and the structure of the algebra (1.2)-(1.4) it follows that the shift term of the Cartan form that appears after performing the transformation is due to the object in front of the antisymmetric part of the generators $K^a{}_b$. The same shift term also appears in the transformation of the spin connection. So we may assume

$$\left(e^{-h}\,\partial_\mu\,e^h\right)_{[ab]} \quad = \quad \omega_{\mu a b} \ + \ \Omega_{\mu[ab]}$$

with $\Omega_{\mu[ab]}$ transforming as a tensor. We made the antisymmetry of $\Omega_{\mu a b}$ in the latter two indices explicit since we want to extend the definition of $\Omega_{\mu a b}$ to denote the tensors in front of the symmetric part of $K^a{}_b$, too. Due to the inverse Higgs effect one can put any Cartan form with a homogenous transformation law to zero without affecting physics [11]. This allows one to neglect $\Omega_{\mu a b}$ and to find relations between $\omega_{\mu a b}$ and $(e^{-h}\partial_\mu e^h)_{ab}$ at the same time (see [6]). In the case of M-theory, non-linear realisations led to the proposal of a hidden E_{11} symmetry [12]. E_{11} appeared as the simplest Kac-Moody algebra which contained the nonsimple algebra of G_{11} but without the momentum generator. So far the discussion of the hidden symmetries was limited to the bosonic sector of the supergravity only. One open problem of the E_{11} conjecture is firstly, how to incorporate fermionic degrees of freedom and secondly, what restrictions this places on the corresponding extension of the algebra G_{11}. The best possible answer would be to treat the bosons and the fermions (gravitino) on the same footing, *i.e.* to generate from an extended group $\tilde{G}_{11}$ the covariant derivative of the gravitino accompanied by the fermionic shifts in the bosonic field strengths and the spin connection. Avoiding the construction of a group extension it is -of course- possible to introduce the fermionic shifts in the bosonic field strengths just by hand [13]. But then it remains unclear how the extended bosonic symmetries couple to the fermionic symmetries, *i.e.* what the extended hidden symmetry group actually looks like. Alternatively, one can try to generalise the fields of the theory to superfields in super-space aiming to find the fermionic field equations by twisted superdualities. This was performed for a two dimensional model in [14].

Historically, the identification of the Kac-Moody algebra that describes the hidden symmetries of the theory was performed by an algebra, which did not take the role of the momentum operator as a central charge of the supersymmetry algebra into account. Later the semi-direct product of E_{11} and representations of the momentum generator in eleven dimensions were considered. The semi-direct product includes non-trivial commutators of the momentum generator with the gauge field generators, which close in the central charges of the supersymmetry algebra in eleven dimensions [15]. Of course, the momentum generator itself appears as a central charge of this algebra.

In the following we will rather take the semi-direct product of some low-level generators of E_{11} (*i.e.* G_{11}) when split into representation of $SL(11)$ with a spinor representation of $SO(1,10)$. The Lorentz group can be obtained from the gravity line by using the Cartan involution (or the temporal involution

3

[16]). In this way, we are still free to define the anti-commutation relations of this fermionic generator with itself and we will naturally choose the supersymmetry algebra of the relevant supergravity theory. This algebra contains the momentum generator apart from the central charges, and so the momentum generator can effectively be added via a semi-direct product. The occurrence of a fermionic generator parametrising the coset of a non-linear realisation will result in a covariant expression for a fermionic parameter, and we suggest to identify this covariant expression with the Killing spinor equation of the theory under consideration. This fermionic parameter is, however, not part of the fields of the theory and thus we still keep a purely bosonic background but yet include supersymmetry into the ansatz of (1.1).

Since the spin representation we multiply is connected with the $SO(1, 10)$ subgroup generated by the antisymmetric combination of the generators $K^a{}_b$ of $SL(11)$, we just keep these and throw the symmetric combination away (avoiding topological difficulties [18]). We will partly answer the question as to whether there exists an extension of the algebra of G_{11} by a fermionic generator $Q_{\dot\alpha}$ generating a parameter $\varepsilon^{\dot\alpha}$, so that the Cartan form finally produces a covariant derivative of $\varepsilon^{\dot\alpha}$,

$$\hat{D}_\mu \varepsilon^{\dot\alpha} \;=\; \left(\partial_\mu \varepsilon^{\dot\alpha} - \frac{1}{4}\, \omega_{\mu ab}\, \Gamma^{ab}\, \varepsilon^{\dot\alpha} \right) - \frac{1}{2\cdot 144} \left(\Gamma^{\alpha\beta\gamma\delta}{}_\mu - 8\, \Gamma^{\beta\gamma\delta}\delta^\alpha_\mu \right) \varepsilon^{\dot\alpha}\, G_{\alpha\beta\gamma\delta} \;, \qquad (1.7)$$

identical to the the Killing spinor equation. Algebraically this problem is closely related to the full program of consistently including fermions into a nonlinear realisation and find them to be Goldstonian. The difference is merely that we do not have to consider the fermionic shifts induced on the bosonic fields. One should note the conceptual difference to [2], whose Cartan form contains the potentials $A_{c_1\ldots c_3}$ and $A_{c_1\ldots c_6}$ and not the field strengths $G_{(4)}$ and $F_{(7)}$.

2   Supersymmetrisation

The ansatz for the superalgebra is mainly fixed by the structure of the covariant derivative we want to generate. Nevertheless one has to make some choices. There are several hints contained in the literature as to how the supersymmetric extension might look like [2, 6, 17]. For different reasons all of these three papers had to take a second, "unphysical" spinorial generator $\tilde{Q}_{\dot\alpha}$ into account. We call this generator "unphysical" since we do not identify the operator which arises in front of $\tilde{Q}_{\dot\alpha}$ with the covariant derivative of a physical quantity. In fact, in our calculation we observed the need for a second fermionic generator, too. We will discuss the technical reason below. The most convincing heuristic argument for the second fermionic generator in the approach via nonlinear realisations is derived from the closure of G_{11} with the conformal group. The conformal group in $d = 11$ is isomorphic to $SO(2, 11)$, whose lowest irreducible spin representation is of dimension $2^6 = 2 \cdot 32$, i.e. twice the amount of the spinor representation in $d = 11$. This superalgebra was explicitly constructed in [17].

It was laid out in the introduction that the covariant field strengths, and in particular their antisymmetrisation, are only found after taking the closure with the conformal group. It therefore appears to be natural to include two fermionic generators $Q_{\dot\alpha}$ and $\tilde{Q}_{\dot\alpha}$ and multiply the group element in (1.1) from the right by

$$g_\varepsilon \;=\; e^{\varepsilon^{\dot\alpha}\,(\,Q_{\dot\alpha} + \tilde{Q}_{\dot\alpha}\,)} \;. \qquad (2.1)$$

The new Cartan form becomes:

$$\mathcal{A} \;=\; g_\varepsilon^{-1}\, d\, g_\varepsilon \;+\; g_\varepsilon^{-1} \left(g_B^{-1}\, d\, g_B \right) g_\varepsilon \;.$$

We work out the first term on the right hand side using eq. (B.1) ending up with an expansion of the form

$$g_\varepsilon^{-1}\, d\,g_\varepsilon \;\; = \;\; d\varepsilon^{\dot\alpha}\,(\,Q_{\dot\alpha} + \tilde{Q}_{\dot\alpha}\,) \; - \; \frac{1}{2}\,[\,\varepsilon^{\dot\alpha}\,(\,Q_{\dot\alpha} + \tilde{Q}_{\dot\alpha}\,),\, d\varepsilon^{\dot\beta}\,(\,Q_{\dot\beta} + \tilde{Q}_{\dot\beta}\,)\,] \; + \; \dots$$

where we have only expanded up to second order since we will soon find that due to the Jacobi identities commutators with more than two fermionic generators vanish (see remark 1 on page 14). For the second bit in the expression $\mathcal{A}$ we introduce a shorthand notation. We set $g_B^{-1}\, d\,g_B \,=\, \sum_{i=1}^{4}\,(\dots)\,\mathfrak{G}_i$ with $\mathfrak{G}_i \in \{\, P_a,\, K^a{}_b,\, R^{c_1 c_2 c_3},\, R^{c_1 \dots c_6}\,\}$ where the dots in brackets refer to the prefactors in eq. (1.5) determined in the last paragraph. Then it reads

$$g_\varepsilon^{-1}\,\left(g_B^{-1}\, d\,g_B\right)\, g_\varepsilon \;\; = \;\; g_\varepsilon^{-1}\,\left(\sum_{i=1}^{4}(\dots)\,\mathfrak{G}_i\right)\, g_\varepsilon$$

and using eq. (B.2) we obtain for each of the four individual contributions an expansion of the type

$$g_\varepsilon^{-1}\mathfrak{G}_i\,g_\varepsilon \;\; = \;\; \mathfrak{G}_i \; - \; [\,\varepsilon^{\dot\alpha}\,(\,Q_{\dot\alpha} + \tilde{Q}_{\dot\alpha}\,),\, \mathfrak{G}_i\,] \; + \; \frac{1}{2!}\,[\,\varepsilon^{\dot\beta}\,(\,Q_{\dot\beta} + \tilde{Q}_{\dot\beta}\,),\, [\,\varepsilon^{\dot\alpha}\,(\,Q_{\dot\alpha} + \tilde{Q}_{\dot\alpha}\,),\, \mathfrak{G}_i\,]\,] \; + \; \dots$$

It is feasible to organise the expansions by the power in the fermionic parameter $\varepsilon^{\dot\alpha}$ they contain, *i.e.*

$$\mathcal{A} \;\; = \;\; \sum_{k=0}^{\infty} \mathcal{A}^{(k)}, \qquad \mathcal{A}^{(0)} \;=\; (g_B^{-1}\, d\,g_B)\,.$$

At zeroth order we just recover the purely bosonic elements of the Cartan form. Formally, the expansions goes all the way up to infinity depending on our choice of (anti)commutation relations of the fermionic generators. For the super algebra we are going to use it will terminate at $k = 3$.

2.1 Linearised Analysis, i.e. $\mathcal{O}(\varepsilon^2)$

To first order in $\varepsilon^{\dot\alpha}$ the Cartan form looks like

$$\mathcal{A}^{(1)} \;\; = \;\; d\varepsilon^{\dot\alpha}\,(\,Q_{\dot\alpha} + \tilde{Q}_{\dot\alpha}\,) \; - \; \varepsilon^{\dot\alpha}\sum_{i=1}^{4}[\,(\,Q_{\dot\alpha} + \tilde{Q}_{\dot\alpha}\,),\,(\dots)\,\mathfrak{G}_i\,]$$

Now we evaluate the terms linear in ε step by step. Because of the $\mathbb{Z}_2$-grading of a superalgebra, the commutators between fermionic and bosonic generators can only yield fermionic generators. They additionally have to fulfil the super Jacobi identities. We set [*2]

$$
\begin{aligned}
[\,Q_{\dot\alpha},\, K^{[bc]}\,] &= \;\; (k^{[bc]})_{\dot\alpha}{}^{\dot\beta}\, Q_{\dot\beta} & [\,\tilde{Q}_{\dot\alpha},\, K^{[bc]}\,] &= \;\; (k^{[bc]})_{\dot\alpha}{}^{\dot\beta}\, \tilde{Q}_{\dot\beta} \\
[\,Q_{\dot\alpha},\, R^{c_1 c_2 c_3}\,] &= \;\; \delta\,(\Gamma^{c_1 c_2 c_3})_{\dot\alpha}{}^{\dot\beta}\, \tilde{Q}_{\dot\beta} & [\,\tilde{Q}_{\dot\alpha},\, R^{c_1 c_2 c_3}\,] &= \;\; \kappa\,(\Gamma^{c_1 c_2 c_3})_{\dot\alpha}{}^{\dot\beta}\, Q_{\dot\beta} & (2.2)\\
[\,Q_{\dot\alpha},\, R^{c_1 \dots c_6}\,] &= \;\; \frac{2\delta\kappa}{c_{3,3}}\,(\Gamma^{c_1 \dots c_6})_{\dot\alpha}{}^{\dot\beta}\, Q_{\dot\beta} & [\,\tilde{Q}_{\dot\alpha},\, R^{c_1 \dots c_6}\,] &= \;\; \frac{2\delta\kappa}{c_{3,3}}\,(\Gamma^{c_1 \dots c_6})_{\dot\alpha}{}^{\dot\beta}\, \tilde{Q}_{\dot\beta}
\end{aligned}
$$

[*2] Lorentz generator $J^{ab} \,=\, 2 \cdot K^{[ab]} \;\Rightarrow\; k^{[ab]} \,=\, \frac{1}{4}\Gamma^{ab}$.

5

where δ and κ are free parameters. Appendix A shows that this choice is consistent with the super Jacobis. We note that it is essential to observe that in the second line the commutator with $R^{a_1 a_2 a_3}$ exchanges the two different Q_α generators! However, since one of them is non-physical, we only display terms proportional to $Q_{\dot\alpha}$ which look

$$
\begin{aligned}
\mathcal{A}^{(1)} \;=\; dx^\mu \Big\{ \; & \partial_\mu \varepsilon^{\dot\alpha}\, Q_{\dot\alpha} \; - \; \varepsilon^{\dot\alpha}\, e_\mu^b \underbrace{[\, Q_{\dot\alpha},\, P_b \,]}_{0\,!} \; - \; \varepsilon^{\dot\alpha}\, \omega_{\mu bc}\, [\, Q_{\dot\alpha},\, K^{[bc]} \,] \\[2mm]
& - \; \varepsilon^{\dot\alpha}\, \frac{1}{4!}\,(4\,\tilde D_\mu A_{c_1 c_2 c_3})\,[\, \tilde Q_{\dot\alpha},\, R^{c_1 c_2 c_3} \,] \; - \; \varepsilon^{\dot\alpha}\, \frac{1}{7!}\,(7\,\tilde D_\mu A_{c_1 \ldots c_6})\,[\, Q_{\dot\alpha},\, R^{c_1 \ldots c_6} \,] \Big\}
\end{aligned}
\tag{2.3}
$$

and the commutator $[Q, P]$ vanishes due to the Jacobi identity Nr. (14) in Appendix A. The further simplifications of eq. (2.3) are straightforward. The only trick one has to keep in mind is connected with rewriting the term containing the generator $R^{c_1 \ldots c_6}$. We have to use the equations of motion, *i.e.* the condition that the two gauge field strengths $G_{(4)}$ and $F_{(7)}$ are related by Hodge duality eq. (1.6), to draw the following conclusion:[*3]:

$$
\frac{1}{6!}\, F_{\mu \mu_2 \ldots \mu_7}\, \Gamma^{\mu_2 \ldots \mu_7} \;=\; \frac{1}{4!}\, \Gamma_\mu{}^{\beta_1 \ldots \beta_4}\, G_{\beta_1 \ldots \beta_4}
\tag{2.4}
$$

Using this identity we may rewrite the contribution delivered by the 6-form potential into:

$$
\varepsilon^{\dot\alpha}\, \frac{1}{7!}\,(7\,\tilde D_\mu A_{c_1 \ldots c_6})\,[\, Q_{\dot\alpha},\, R^{c_1 \ldots c_6} \,] \;=\; \frac{2\delta\kappa}{c_{3,3}}\, \varepsilon^{\dot\alpha} \left\{ \frac{1}{7 \cdot 4!}\, \Gamma_\mu{}^{\beta_1 \ldots \beta_4}\, G_{\beta_1 \ldots \beta_4} \right\} Q_{\dot\beta}
$$

Inserting this into eq. (2.3) one obtains:

$$
\begin{aligned}
\mathcal{A}^{(1)} \;=\; dx^\mu \Big\{ \; & \partial_\mu \varepsilon^{\dot\alpha} \; - \; \omega_{\mu bc}\,(k^{[bc]})_{\dot\beta}{}^{\dot\alpha}\, \varepsilon^{\dot\beta} \\[2mm]
& - \; \frac{1}{4!} \left(\left[\frac{2\delta\kappa}{7\,c_{3,3}} \right] (\Gamma^{c_0 \ldots c_3}{}_\mu)_{\dot\beta}{}^{\dot\alpha} \; + \; [\,\kappa\,]\, \delta_\mu^{c_0}\, (\Gamma^{c_1 c_2 c_3})_{\dot\beta}{}^{\dot\alpha} \right) G_{c_0 c_1 c_2 c_3}\, \varepsilon^{\dot\beta} \Big\} Q_{\dot\alpha}
\end{aligned}
\tag{2.5}
$$

Up to this point we have not fixed any of the free parameters appearing in our predictions for the equations of motion of 11d supergravity. Now we want to fix the three free parameters $c_{3,3}$, δ and κ in a way, which finally produce the correct equation of motion for the gauge field strength $G_{(4)}$ and the Killing spinor equation. We have to choose

$$
c_{3,3} \;=\; 1, \qquad \frac{2\delta\kappa}{7\,c_{3,3}} \;=\; \frac{1}{12}, \qquad \kappa \;=\; -\,\frac{8}{12}\,.
\tag{2.6}
$$

These constraints lead to $\delta = -\frac{7}{16}$. The complete Cartan form for the fermionic generator becomes

$$
\mathcal{A}^{(1)} \;=\; dx^\mu\, (\hat D_\mu^{(0)} \varepsilon)^{\dot\alpha} Q_{\dot\alpha} \; + \; dx^\mu\, (\hat\Delta_\mu^{(0)} \varepsilon)^{\dot\alpha} \tilde Q_{\dot\alpha}
$$

with $(D_\mu^{(0)} \varepsilon)^{\dot\alpha}$ the Killing spinor equation of eq. (1.7) and $(\Delta_\mu^{(0)} \varepsilon)^{\dot\alpha}$ the operator in front of the "unphysical" generator $\tilde Q_{\dot\alpha}$ which is of no importance for the physical quantities.

[*3] $\Gamma_{a_1 \ldots a_j} = \dfrac{(-1)^{\frac{(11-j)(11-j-1)}{2}}}{(11-j)!}\, \Gamma_{a_1 \ldots a_{11}}\, \Gamma^{a_{j+1} \ldots a_{11}}$ and $\Gamma^{0 \ldots 10} = \mathrm{sgn}\{0 \ldots 10\}$

The higher order corrections can be computed similarly. At next order $\mathcal{O}(\varepsilon^3)$ we obtain

$$
\begin{aligned}
\mathcal{A}^{(2)} \;=\; & -\frac{1}{2}\,\varepsilon^{\dot\alpha}\,(\hat D^{(0)}_\mu\varepsilon)^{\dot\beta}\,\{\,Q_{\dot\alpha},\,Q_{\dot\beta}\,\}\,dx^\mu \;-\; \frac{1}{2}\,\varepsilon^{\dot\alpha}\,(\hat\Delta^{(0)}_\mu\varepsilon)^{\dot\beta}\,\{\,\tilde Q_{\dot\alpha},\,\tilde Q_{\dot\beta}\,\}\,dx^\mu \\
& -\frac{1}{2}\,\varepsilon^{\dot\alpha}\,\left((\hat D^{(0)}_\mu\varepsilon)^{\dot\beta}\,\{\,\tilde Q_{\dot\alpha},\,Q_{\dot\beta}\,\} \;+\; (\hat\Delta^{(0)}_\mu\varepsilon)^{\dot\beta}\,\{\,Q_{\dot\alpha},\,\tilde Q_{\dot\beta}\,\}\right)\,dx^\mu \ .
\end{aligned}
\tag{2.7}
$$

which is an expression in the various central charges (cf. (A.6)). All terms $\mathcal{A}^{(k>2)}$ vanish due to the Jacobi identities (A.14)-(A.17) and remark 1 on page 14. Putting all the results for $\mathcal{A}$ together one obtains

$$
\begin{aligned}
\mathcal{A} \;=\; \sum_{i=1}^{\infty}\mathcal{A}^{(i)} \;=\; & (g_B^{-1}dg_B) \;+\; dx^\mu\,(\hat D^{(0)}_\mu\varepsilon)^{\dot\alpha}\left(Q_{\dot\alpha} \;-\; \frac{1}{2}\,\varepsilon^{\dot\beta}\,\{\,Q_{\dot\alpha},\,Q_{\dot\beta}\,\}\right) \\
& +\; dx^\mu\,(\hat\Delta^{(0)}_\mu\varepsilon)^{\dot\alpha}\left(\tilde Q_{\dot\alpha} \;-\; \frac{1}{2}\,\varepsilon^{\dot\beta}\,\{\,\tilde Q_{\dot\alpha},\,\tilde Q_{\dot\beta}\,\}\right) \\
& -\frac{1}{2}\,\varepsilon^{\dot\alpha}\left((\hat D^{(0)}_\mu\varepsilon)^{\dot\beta}\,\{\,\tilde Q_{\dot\alpha},\,Q_{\dot\beta}\,\} \;+\; (\hat\Delta^{(0)}_\mu\varepsilon)^{\dot\beta}\,\{\,Q_{\dot\alpha},\,\tilde Q_{\dot\beta}\,\}\right)\,dx^\mu \ .
\end{aligned}
\tag{2.8}
$$

It is important to notice that there is a correction term to the bosonic vielbein in front of the momentum generator coming from the anticommutator $\{Q,Q\}$, which is proportional to $\hat D^{(0)}_\mu\varepsilon$. If one imposes the Killing spinor equation the bosonic vielbein is left unchanged. This is an a posteriori argument for the identification of the Killing spinor equation and our notion of "physical" and "unphysical" fermionic generators.

3 $N=1$ pure supergravity

A similar construction as the one before can also be used to find the Killing spinor equation of $N=1$ pure supergravity in ten dimensions [19][*4]. The group that was used to construct the covariant objects of the bosonic sector of the theory was spelled out in [9], and was called G_I. The group element is taken to be

$$
g = e^{x^\mu P_\mu}\,e^{h_a{}^b K^a{}_b}\,e^{\frac{1}{8!}A_{a_1\ldots a_8}R^{a_1\ldots a_8}}\,e^{\frac{1}{6!}A_{a_1\ldots a_6}R^{a_1\ldots a_6}}\,e^{\frac{1}{2!}A_{a_1 a_2}R^{a_1 a_2}}\,e^{A\,R}
\tag{3.1}
$$

and the commutators of the generators satisfy relations analogous to eq. (1.2)-(1.3) but with a new set of gauge field commutators, whose algebra is given by

$$
[\,R,\,R^{a_1\ldots a_p}\,] \;=\; c_p\,R^{a_1\ldots a_p}, \quad [\,R^{a_1 a_2},\,R^{a_3\ldots a_8}\,] \;=\; c_{2,6}\,R^{a_1\ldots a_8}, \qquad c_2 = -c_6 = c_{2,6} = \frac{1}{2} \ .
\tag{3.2}
$$

The corresponding field strengths (closure with the conformal group yields antisymmetric tensors as described before) are

$$
F_{a_1} \;=\; \partial_{a_1}A
\tag{3.3}
$$

$$
F_{a_1 a_2 a_3} \;=\; e^{-\frac{A}{2}}\left(3\,\partial_{[a_1}A_{a_2 a_3]}\right)
\tag{3.4}
$$

$$
F_{a_1\ldots a_7} \;=\; e^{\frac{A}{2}}\left(7\,\partial_{[a_1}A_{a_2\ldots a_7]}\right)
\tag{3.5}
$$

$$
F_{a_1\ldots a_9} \;=\; 9\left(\partial_{[a_1}A_{a_2\ldots a_9]} - 7\cdot 2\,A_{[a_1 a_2}\partial_{a_3}A_{a_4\ldots a_9]}\right)
\tag{3.6}
$$

[*4] The differences in the conventions to [19] are the same as described in footnote [*1].

with the two first order equations of motion

$$*F^{(3)} = F^{(7)} \ , \qquad *F^{(1)} = F^{(9)} \ . \tag{3.7}$$

In our notation the two Killing spinor equations read

$$\delta\psi_\mu \;=\; D_\mu\varepsilon \;+\; \frac{1}{72}\left(\Gamma^{\nu\rho\sigma}{}_\mu - 9\delta^\nu_\mu\,\Gamma^{\rho\sigma}\right)\varepsilon\,F_{\nu\rho\sigma} \tag{3.8}$$

$$\delta\chi \;=\; \sqrt{\frac{1}{8}}\left(F_\mu\,\Gamma^\mu\,\varepsilon \;-\; \frac{1}{12}\,\Gamma^{\nu\rho\sigma}\,F_{\nu\rho\sigma}\,\varepsilon\right) \tag{3.9}$$

As in the case of eleven dimensional supergravity treated previously we enhance the group element by a fermionic generator $\exp(\varepsilon^\alpha\,Q_\alpha)$ from the right. The extension of the algebra is defined by commutation relations similarly to (2.2) and reads

$$[Q_{\dot\alpha},\, R^{a_1 a_2}] = s_2(\Gamma^{a_1 a_2})_{\dot\alpha}{}^{\dot\beta} Q_{\dot\beta}$$

$$[Q_{\dot\alpha},\, R^{a_1 \cdots a_6}] = s_6(\Gamma^{a_1 \cdots a_6})_{\dot\alpha}{}^{\dot\beta} Q_{\dot\beta} + q_6(\Gamma^{a_1 \cdots a_6})_{\dot\alpha}{}^{\dot\beta}\tilde{Q}_{\dot\beta} \tag{3.10}$$

$$[Q_{\dot\alpha},\, R^{a_1 \cdots a_8}] = q_8(\Gamma^{a_1 \cdots a_8})_{\dot\alpha}{}^{\dot\beta}\tilde{Q}_{\dot\beta} \ .$$

Since previous experience has taught us to introduce a second fermionic generator we do it here again and discuss the reason later.

In eq. (3.10) we have not written down commutators of the dilaton generator with the supercharges. Actually, by checking the Jacobi identities it is found that the super extension is inconsistent with the interpretation of the dilaton generator as an element of the Cartan subalgebra. We are not surprised. The origin of this problem is connected to the difficulties with the symmetric part of $K^a{}_b$. In the footnote on page 5 we have used the antisymmetric Γ-matrices to parameterise the antisymmetric part of the K^{ab} generators. The dilaton generator R can be understood from an eleven dimensional perspective as a generator built from the trace parts of the eleven dimensional $K^a{}_b$. Since we have also realised the gauge generators inside the Clifford algebra there is no algebraic possibility to realise the R generator inside the Clifford algebra at the same time. Perhaps there is another mathematical technique to get rid of the dilaton but it is unknown to us. Since we do not need the contributions from the dilaton generator R, we have dropped it by hand; we have to keep in mind though that finding a closing algebra including R needs further consideration.

As usually, we are mainly interested in those elements that close in the untilded Q_α. The various constants s_i and q_i are not linearly independent but have to be chosen such that the Jacobi identities close. We have to formally define also commutators

$$[\tilde{Q}_{\dot\alpha},\, R^{a_1 a_2}] = \tilde{q}_2(\Gamma^{a_1 a_2})_{\dot\alpha}{}^{\dot\beta}\tilde{Q}_{\dot\beta}$$

$$[\tilde{Q}_{\dot\alpha},\, R^{a_1 \cdots a_6}] = \tilde{s}_6(\Gamma^{a_1 \cdots a_6})_{\dot\alpha}{}^{\dot\beta} Q_{\dot\beta} + \tilde{q}_6(\Gamma^{a_1 \cdots a_6})_{\dot\alpha}{}^{\dot\beta}\tilde{Q}_{\dot\beta} \tag{3.11}$$

$$[\tilde{Q}_{\dot\alpha},\, R^{a_1 \cdots a_8}] = \tilde{s}_8(\Gamma^{a_1 \cdots a_8})_{\dot\alpha}{}^{\dot\beta} Q_{\dot\beta} \ .$$

The Jacobi identities put the constraint below on the free coefficients above:

$$[Q_\alpha,\, [R^p,\, R^q]] \;\;= [[Q_\alpha,\, R^p],\, R^q] \;+\; [R^p,\, [Q_\alpha,\, R^q]] \tag{3.12}$$

and the one with Q and $\tilde{Q}$ exchanged. We note that in this case the commutation relations are actually carried by the gamma-matrices if we totally antisymmetrise the indices. Evaluated on the totally antisymmetric part of the Clifford algebra it gives the following constraints:

(p,q)	$[\,Q_\alpha,\,[\,R^p,\,R^q\,]\,]$ - constraint
(2,2)	none
(2,6)	$c_{2,6}\,q_8 = q_6\,(\,s_2 - \tilde{q}_2\,)$
(2,8)	$0 = q_8\,(s_2 - \tilde{q}_2)$
(6,6)	none
(6,8)	$0 = q_8\,(s_6 - \tilde{q}_6)$ & $0 = q_6\tilde{s}_6\,(s_2 - \tilde{q}_2)$
(8,8)	none

This must be solved by setting

$$ s_2 \;=\; \tilde{q}_2\,, \qquad q_8 \;=\; 0 \tag{3.13} $$

The constraints from the $[\,\tilde{Q}_\alpha,\,[\,R^p,\,R^q\,]\,]$ Jacobi identity are completely analogous. The remaining two constants s_2, s_6 are unconstrained from this point of view and can be chosen as to generate the Killing spinor equation of pure supergravity in ten dimensions. The Cartan form becomes

$$
g^{-1}\,d\,g \;=\; g_B^{-1}dg_B \;+\; dx^a\left(\partial_a\varepsilon^\beta - \frac{s_2}{3!}\,F_{aa_2a_3}\,(\Gamma^{a_2a_3})_\alpha{}^\beta\,\varepsilon^\alpha - \frac{s_6}{7!}\,F_{aa_2\cdots a_7}\,(\Gamma^{a_2\cdots a_7})_\alpha{}^\beta\,\varepsilon^\alpha\right)Q_\beta
$$
$$
+\, dx^a\left(-\frac{q_6}{7!}\,F_{aa_2\cdots a_7}\,(\Gamma^{a_2\cdots a_7})_\alpha{}^\beta\,\varepsilon^\alpha - \frac{q_8}{9!}\,F_{aa_2\cdots a_9}\,(\Gamma^{a_2\cdots a_9})_\alpha{}^\beta\,\varepsilon^\alpha\right)\tilde{Q}_\beta + \dots \;. \tag{3.14}
$$

Using the equations of motion of eq. (3.7) as in the last section to get rid of the dual field strengths[*5],

$$
\frac{1}{p!}\,F_{c_0c_1\dots c_p}\,\Gamma^{c_1\dots c_p} \;=\; \frac{(-1)^p\,(-1)^{\frac{(10-p)(9-p)}{2}}}{(10-p-1)!}\,\Gamma_{11}\,G^{a_{p+1}\dots a_9}\,\Gamma_{c_0 a_{p+1}\dots a_9}\;,
$$

the expression in front of Q_β in (3.14) simplifies to

$$
\partial_a\varepsilon^\beta \;+\; \frac{1}{72}\left(\frac{12\,s_6}{7}\,(\Gamma^{a_1a_2a_3}{}_\mu)_\alpha{}^\beta - 12\,s_2\,\delta^{a_1}_a\,(\Gamma^{a_2a_3})_\alpha{}^\beta\right)\varepsilon^\alpha\,F_{a_1a_2a_3}
$$

Comparison with the Killing spinor equation eq. (3.8) fixes

$$ s_6 \;=\; \frac{7}{12},\quad s_2 \;=\; \frac{3}{4}\,. \tag{3.15} $$

Finally, let us take a look at the expression that builds up in front of $\tilde{Q}$ in eq. (3.14). After using the equations of motion eq. (3.7) we found indications that this could be connected with the algebraic Killing spinor equation eq. (3.9). If one contracts this expression with Γ^a one obtains

$$ \text{in } \tilde{Q}\,: \quad -\frac{q_6}{3!}(\Gamma^{b_1b_2b_3})_\alpha{}^\beta\,\varepsilon^\alpha\,F_{b_1b_2b_3} \;+\; q_8\,F_b(\Gamma^b)_\alpha{}^\beta\,\varepsilon^\alpha \tag{3.16} $$

[*5] $\Gamma_{a_1\dots a_j} = \frac{1}{(10-j)!}\,\Gamma_{a_1\dots a_j b_1\dots b_{10-j}}\,\Gamma^{b_{10-j}\dots b_1}$

which is exactly the expected structure. Due to the gauge algebra of G_I in (3.2) which requires $c_{2,8} \equiv 0$ we have to place the stopper $q_8 = 0$ (see table) which deletes the second term of the above equation. In D_8^{+++}, however, $c_{2,8} \equiv 0$ is not required anymore, and thus $q_8 \neq 0$ is possible [20]. Obviously the solution to the dilaton puzzle holds the key to the completion of the picture.

The relation of this super extension to the one defined in the case of eleven dimensional supergravity is not understood.

4 Conclusions

We have shown that part of the original G_{11} group used in [6] to define M-theory as a nonlinear realisation possesses an extension whose Cartan form produces a covariant derivative of spinors identical to the Killing spinor equation of eleven dimensional supergravity. It is appealing that the structure of the Killing spinor equation is inevitably generated by the group structure. On the other hand this method is not yet expected to give the full super covariant objects like the supercovariant field strengths including the fermionic shifts.

A similar construction was used for $N = 1$ pure supergravity but runs into difficulties because of the dilaton generator that arises by dimensional reduction from eleven dimensions on a torus.

It would be interesting to see how dimensional reduction can be made consistent with a super algebra, since the problems with the dilaton generator are generic. We expect that the approach presented here can be straightforwardly applied to other supergravity theories and other low-level expansions of very-extended Lie algebras (see [20]) and might hold some clues about the above stated problem. In this case it could be helpful to classify all maximal supersymmetric version of supergravities [21, 22] but also other solutions that preserve different amounts of supersymmetry.

A crucial point turned out to be the need to introduce at least two fermionic generators. Working with just one generator it is not possible to fix the free parameters of eq. (2.6) consistently. The doubling of the fermionic generators is interesting of itself and related to the identification of positive and negative roots of a super algebra.

Finally, it would be useful to see the relation to other approaches assuming infinite dimensional Kac-Moody algebras as symmetry algebras of (super)gravity theories [23, 24]. Our approach of taking the semi-direct product with a spinor representation should be applicable to these models, too.

5 Acknowledgements

A Jacobi Identities of M-Theory

In this appendix we list for completeness the set of Jacobi identities in the case of eleven dimensional supergravity. The Lie-bracket must be understood according to the parity $(|\ldots|)$ of the generators as commutators and anticommutators, respectively, *i.e.*

$$[X, Y] = -(-1)^{|X| \cdot |Y|} \cdot [Y, X] .$$

The corresponding Jacobi identity reads

$$[X, [Y, Z]] = [[X, Y], Z] + (-1)^{|X| \cdot |Y|} [Y, [X, Z]] . \tag{A.1}$$

The consistency of the purely bosonic generators was established in [6]. So we concentrate on the Jacobis containing one or more fermionic generators. In the next section we consider Jacobis containing at most one fermionic generator. We have chosen $Q_{\dot\alpha}$ but the case of $\tilde{Q}_{\dot\alpha}$ is totally symmetric.

A.1 Jacobis with one fermionic generator $Q_{\dot\alpha}$

For commutators of two $R^{c_1 \cdots c_3}$ given in eq. (1.4) we make explicit the implicit requirement on the symmetry by introducing the anti symmetrisation symbol on the right hand side:

$$[R^{[c_1 \ldots c_3}, R^{d_1 \ldots d_3]}] = c_{3,3} R^{[c_1 \ldots c_3 d_1 \ldots d_3]} . \tag{A.2}$$

The additional projection onto the totally antisymmetric part makes the fermionic extension possible, *i.e.* the above algebra can be realised as the total antisymmetric part in the Clifford multiplication. In contrast to eq. (2.2) we define for the purpose of shortness

$$[Q_{\dot\alpha}, K^c{}_d] = (k^c{}_d)_{\dot\alpha}{}^{\dot\beta} Q_{\dot\beta} \tag{A.3}$$

$$[Q_{\dot\alpha}, R^{c_1 c_2 c_3}] = \delta (\Gamma^{c_1 c_2 c_3})_{\dot\alpha}{}^{\dot\beta} Q_{\dot\beta} . \tag{A.4}$$

The reader may immediately notice that we seem to work with the symmetric part of $K^a{}_b$, too. This is correct up to the fact that we don't know an explicit realisation of these $k^c{}_d$ (if it exists at all [18]). To resolve any ambiguities one can restrict the corresponding equation onto the antisymmetric part of $k^c{}_d$, which possess an explicit realization as the generators of $\mathfrak{spin}(1, 10)$. In Tab. 2 we list all Jacobi identities containing at most one fermionic generator by displaying the left hand side of eq. (A.1).

Nr.	l.h.s. of eq. (A.1)	satisfied ?
(1)	$[Q_{\dot\alpha}, [P_b, P_c]]$	trivial
(2)	$[Q_{\dot\alpha}, [K^b{}_c, K^d{}_e]]$	cf. proof
(3)	$[Q_{\dot\alpha}, [R^{b_1 b_2 b_3}, R^{c_1 c_2 c_3}]]$	cf. proof
(4)	$[Q_{\dot\alpha}, [R^{b_1 \ldots b_6}, R^{c_1 \ldots c_6}]]$	cf. proof
(5)	$[Q_{\dot\alpha}, [P_b, K^c{}_d]]$	trivial
(6)	$[Q_{\dot\alpha}, [P_b, R^{c_1 c_2 c_3}]]$	trivial
(7)	$[Q_{\dot\alpha}, [P_b, R^{c_1 \ldots c_6}]]$	trivial
(8)	$[Q_{\dot\alpha}, [K^b{}_c, R^{c_1 c_2 c_3}]]$	cf. proof
(9)	$[Q_{\dot\alpha}, [K^b{}_c, R^{c_1 \ldots c_6}]]$	cf. proof
(10)	$[Q_{\dot\alpha}, [R^{b_1 b_2 b_3}, R^{c_1 \ldots c_6}]]$	cf. proof

Tab. 2 One fermionic $Q_{\dot\alpha}$-generator

Proof. of Nr. (2)

$$
\begin{aligned}
[\, Q_{\dot\alpha}, \, [\, K^b{}_c, \, K^d{}_e \,]\,] &= [\, Q_{\dot\alpha}, \, \delta^d_c \, K^b{}_e \, - \, \delta^b_e \, K^d{}_c \,]\,] \\
&= \left\{ \delta^d_c \, (k^b{}_e)_{\dot\alpha}{}^{\dot\beta} \, - \, \delta^b_e \, (k^d{}_c)_{\dot\alpha}{}^{\dot\beta} \right\} \, Q_{\dot\beta}
\end{aligned}
$$

$$
\begin{aligned}
[\, Q_{\dot\alpha}, \, [\, K^b{}_c, \, K^d{}_e \,]\,] &= [\,[\, Q_{\dot\alpha}, \, K^b{}_c \,], \, K^d{}_e \,]\,] \, + \, [\, K^b{}_c, \, [\, Q_{\dot\alpha}, \, K^d{}_e \,]\,] \\
&= (\,[\, k^b{}_c, \, k^d{}_e \,])_{\dot\alpha}{}^{\dot\gamma} \, Q_{\dot\gamma}
\end{aligned}
$$

i.e.

$$
(\,[\, k^b{}_c, \, k^d{}_e \,])_{\dot\alpha}{}^{\dot\beta} = \delta^d_c \, (k^b{}_e)_{\dot\alpha}{}^{\dot\beta} \, - \, \delta^b_e \, (k^d{}_c)_{\dot\alpha}{}^{\dot\beta}
$$

forms a representation of $K^a{}_b$. $\qquad\qquad\square$

Proof. of Nr. (3)

Using eq. (1.4) we get

$$
\begin{aligned}
[\, Q_{\dot\alpha}, \, [\, R^{[c_1 c_2 c_3}, \, R^{c_4 c_5 c_6]} \,]\,] &= [\,[\, Q_{\dot\alpha}, \, R^{[c_1 c_2 c_3} \,], \, R^{c_4 c_5 c_6]} \,]\,] \, + \, [\, R^{[c_1 c_2 c_3}, \, [\, Q_{\dot\alpha}, \, R^{c_4 c_5 c_6]} \,]\,] \\[2mm]
&= \delta\kappa \, \left([\, \Gamma^{[c_1 c_2 c_3}, \, \Gamma^{c_4 c_5 c_6]} \,] \right)_{\dot\alpha}{}^{\dot\gamma} \, Q_{\dot\gamma} \\
&= 2\,\delta\kappa \, (\, \Gamma^{c_1 c_2 c_3 c_4 c_5 c_6} \,)_{\dot\alpha}{}^{\dot\gamma} \, Q_{\dot\gamma}
\end{aligned}
$$

which can be seen as the definition for the action of $Q_{\dot\alpha}$ on $R^{c_1 \cdots c_6}$:

$$
[\, Q_{\dot\alpha}, \, R^{c_1 \cdots c_6} \,] = \frac{2\,\delta\kappa}{c_{3,3}} \, (\, \Gamma^{c_1 c_2 c_3 c_4 c_5 c_6} \,)_{\dot\alpha}{}^{\dot\gamma} \, Q_{\dot\gamma} \tag{A.5}
$$

$$\qquad\qquad\square$$

Proof. of Nr. (4)

By the same reasoning as in the proof of Nr. (3) we find

$$
[\, Q_{\dot\alpha}, \, [\, R^{[c_1 \cdots c_6}, \, R^{d_1 \cdots d_6]} \,]\,] = 2 \left(\frac{2\delta\kappa}{c_{3,3}} \right)^2 \left(\Gamma^{[c_1 \cdots c_6 d_1 \cdots d_6]} \right)_{\dot\alpha}{}^{\dot\gamma} \, Q_{\dot\gamma} \overset{!}{=} 0.
$$

The rhs vanishes due to the anti symmetrisation of 12 out of 11 indices. The lhs vanishes due to the bosonic algebra. $\qquad\square$

Proof. of Nr. (8)

$$
[\, Q_{\dot\alpha}, \, [\, K^c{}_d, \, R^{c_1 c_2 c_3} \,]\,] = [\,[\, Q_{\dot\alpha}, \, K^c{}_d \,], \, R^{c_1 c_2 c_3} \,] \, + \, [\, K^c{}_d, \, [\, Q_{\dot\alpha}, \, R^{c_1 c_2 c_3} \,]\,]
$$

$$\text{lhs} \quad = \quad [\, Q_{\dot{\alpha}}, \, \delta_d^{c_1} \, R^{cc_2c_3} \, + \, \delta_d^{c_2} \, R^{c_1cc_3} \, + \, \delta_d^{c_3} \, R^{c_1c_2c} \,]$$

$$= \quad \delta \, \{\, \delta_d^{c_1} \, \Gamma^{cc_2c_3} \, + \, \delta_d^{c_2} \, \Gamma^{c_1cc_3} \, + \, \delta_d^{c_3} \, \Gamma^{c_1c_2c} \,\}_{\dot{\alpha}}{}^{\dot{\beta}} \, \tilde{Q}_{\dot{\beta}}$$

$$\text{rhs} \quad = \quad \delta \, ([\, k^c{}_d, \, \Gamma^{c_1c_2c_3} \,])_{\dot{\alpha}}{}^{\dot{\beta}} \, \tilde{Q}_{\dot{\beta}}$$

Together[*6]

$$[\, k^c{}_d, \, \Gamma^{c_1c_2c_3} \,] \quad = \quad \delta_d^{p_1} \, \Gamma^{cp_2p_3} \, + \, \delta_d^{p_2} \, \Gamma^{p_1cp_3} \, + \, \delta_d^{p_3} \, \Gamma^{p_1p_2c}$$

$\square$

Proof. of Nr. (9)

Completely analogous to Nr. (8). $\qquad\qquad$ $\square$

Proof. of Nr. (10)

Again analogous to the proof of Nr. (4) we find

$$[\, Q_{\dot{\alpha}}, \, [\, R^{[c_1c_2c_3}, \, R^{d_1\ldots d_6]} \,]\,] \quad = \quad \frac{2\,\delta^2\kappa}{c_{3,3}} \, \left([\, \Gamma^{[c_1c_2c_3}, \, \Gamma^{d_1\ldots d_6]} \,] \right)_{\dot{\alpha}}{}^{\dot{\gamma}} \, Q_{\dot{\gamma}} \;\overset{!}{=}\; 0.$$

$\square$

A.2 Jacobis with two fermionic generators

Nr.	l.h.s. of eq. (A.1)
(11)	$[\, \mathfrak{G}, \, [\, Q_{\dot{\alpha}}, \, Q_{\dot{\beta}} \,]\,]$
(12)	$[\, \mathfrak{G}, \, [\, \tilde{Q}_{\dot{\alpha}}, \, Q_{\dot{\beta}} \,]\,]$
(13)	$[\, \mathfrak{G}, \, [\, \tilde{Q}_{\dot{\alpha}}, \, \tilde{Q}_{\dot{\beta}} \,]\,]$

Tab. 3 Two fermionic $Q_{\dot{\alpha}}$-generators

Defines the action of the bosonic generators $\mathfrak{G} \, = \, \{\, K^a{}_b, \, R^{c_1c_2c_3}, \, R^{c_1\ldots c_6} \,\}$ on the "central charges" of the supersymmetry algebras, *i.e.* on $Z_{\dot{\alpha}\dot{\beta}}$, $\tilde{Z}_{\dot{\alpha}\dot{\beta}}$ and $A_{\dot{\alpha}\dot{\beta}}$ defined by

$$\{\, Q_{\dot{\alpha}}, \, Q_{\dot{\beta}} \,\} \quad = \quad Z_{\dot{\alpha}\dot{\beta}}, \qquad \{\, \tilde{Q}_{\dot{\alpha}}, \, \tilde{Q}_{\dot{\beta}} \,\} \; = \; \tilde{Z}_{\dot{\alpha}\dot{\beta}}, \qquad \{\, \tilde{Q}_{\dot{\alpha}}, \, Q_{\dot{\beta}} \,\} \; = \; A_{\dot{\alpha}\dot{\beta}} \qquad (A.6)$$

Proof. of Nr. 11:

We expand $Z_{\dot{\alpha}\dot{\beta}}$ in the Clifford algebra

$$Z_{\dot{\alpha}\dot{\beta}} \quad = \quad \Gamma^c P_c + \frac{1}{2!} \, \Gamma^{c_1c_2} Z_{c_1c_2} \, + \, \frac{1}{5!} \, \Gamma^{c_1\ldots c_5} Z_{c_1\ldots c_5}$$

[*6] The antisymmetric part of $k^{[cd]} = \frac{1}{4}\Gamma^{cd}$ gives the Gamma-matrix identity $[\, \frac{1}{4}\Gamma^{cd}, \, \Gamma_{c_1c_2c_3} \,] = 3 \cdot \delta^{[d}_{[c_1} \, \Gamma^{c]}{}_{c_2c_3]}$

and similar expressions hold for $\tilde{Z}_{\dot\alpha\dot\beta}$ and $A_{\dot\alpha\dot\beta}$. After repeated application of the Jacobi identities and the formula $Tr(\Gamma^{a_i..a_1}\Gamma_{b_1...b_j}) = 32 \cdot \delta^{a_1...a_i}_{b_1...b_j}$ one obtains $e.g.$ for $\mathfrak{G}_1 = P_a$:

$$[\, R^{c_1 c_2 c_5},\, (\Gamma^a)^{\dot\beta\dot\alpha} Z_{\dot\alpha\dot\beta}\,] \quad = \quad \frac{6}{2!}\, \delta\, \eta^{a[c_1}\delta^{c_2 c_3]}_{[b_1 b_2]} \cdot 32 \cdot A^{b_1 b_2}$$

or finally

$$[\, P_a,\, R^{c_1 c_2 c_3}\,] \quad = \quad -\, 6\, \delta\, \eta^{a[c_1} A^{c_1 c_3]}$$

<div align="right">□</div>

A.3   Jacobis with three fermionic generators

Nr.	l.h.s. of eq. (A.1)	satisfied ?
(14)	$[\, Q_{\dot\alpha},\, [\, Q_{\dot\beta},\, Q_{\dot\gamma}\,]\,]$	cf. proof
(15)	$[\, \tilde{Q}_{\dot\alpha},\, [\, \tilde{Q}_{\dot\beta},\, \tilde{Q}_{\dot\gamma}\,]\,]$	cf. proof
(16)	$[\, \tilde{Q}_{\dot\alpha},\, [\, Q_{\dot\beta},\, Q_{\dot\gamma}\,]\,]$	cf. proof
(17)	$[\, Q_{\dot\alpha},\, [\, \tilde{Q}_{\dot\beta},\, \tilde{Q}_{\dot\gamma}\,]\,]$	cf. proof

Tab. 4 Three fermionic $Q_{\dot\alpha}$-generator

Proof. of Nr. (14) and Nr. (15)

Are just the statements, that $Q_{\dot\alpha}$ ($\tilde{Q}_{\dot\alpha}$) act trivial on the generators appearing on the right hand side of the $\{Q_{\dot\beta},\, Q_{\dot\gamma}\}$ ($\{\tilde{Q}_{\dot\beta},\, \tilde{Q}_{\dot\gamma}\}$) anticommutator. □

Proof. of Nr. (16)

$$[\, \tilde{Q}_{\dot\alpha},\, [\, Q_{\dot\beta},\, Q_{\dot\gamma}\,]\,] \quad = \quad [[\, \tilde{Q}_{\dot\alpha},\, Q_{\dot\beta}\,],\, Q_{\dot\gamma}\,] \,-\, [\, Q_{\dot\beta},\, [\, \tilde{Q}_{\dot\alpha},\, Q_{\dot\gamma}\,]\,]$$

$$[\, \tilde{Q}_{\dot\alpha},\, Z_{\dot\beta\dot\gamma}\,] \quad = \quad -2\,[\, Q_{(\dot\beta},\, A_{|\dot\alpha|\dot\gamma)}\,]$$

<div align="right">□</div>

Proof. of Nr. (17)

Completely analogous to proof Nr. (16) and leads to:

$$[\, Q_{\dot\alpha},\, \tilde{Z}_{\dot\beta\dot\gamma}\,] \quad = \quad -2\,[\, \tilde{Q}_{(\dot\beta},\, A_{\dot\gamma)\dot\alpha}\,]$$

<div align="right">□</div>

Remark 1. The action of $Q_{\dot\alpha}$ and $\tilde{Q}_{\dot\alpha}$ on $A_{\beta\gamma}$ fixes the action of $Q_{\dot\alpha}$ and $\tilde{Q}_{\dot\alpha}$ on $Z_{\beta\gamma}$ and $\tilde{Z}_{\beta\gamma}$. It is consistent to set this action to zero. Then all the three algebras $Z_{\beta\gamma}$, $\tilde{Z}_{\beta\gamma}$ and $A_{\beta\gamma}$ are abelian and don't mix with each other.

B   Formulas

$$e^{-A}\, d\, e^{A} \;=\; dA \;-\; \frac{1}{2!}\,[\,A,\,dA\,] \;+\; \frac{1}{3!}\,[\,A,\,[\,A,\,dA\,]\,] \;+\; \dots \tag{B.1}$$

$$e^{-A}\, B\, e^{A} \;=\; B \;-\; [\,A,\,B\,] \;+\; \frac{1}{2!}\,[\,A,\,[\,A,\,B\,]\,] \;+\; \dots \tag{B.2}$$

C   Bibliography

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